A Review of Radiation and Optics

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Abstract

This paper attempts to summarize selected topics in Radiation and Optics. It is, by no means, a complete coverage of the subject matter, and shouldn't be used as such. Instead, it can help as review material once significant familiarity has been achieved with Maxwell's equations, Poynting's Theorem, Potential Functions, Radiation Integrals, Antennas, and Systems-Based Analysis of Optical Devices.

The summary uses notations that are common in Vector Calculus, and relies heavily on theorems from the same subject. Therefore, a keen understanding of Vector Calculus is required before attempting to read it.

On a final note, one should refer to the notes supplied by Dr. Chester Nisteruk of Manhattan College's Electrical and Computer Engineering department in his ELEG 706 - Radiation and Optics course. This summary was prepared in reference to his notes in Spring of 2011.

References

[1] Nisteruk, C., *ELEG 706 - Radiation & Optics Lecture Notes.* Manhattan College, Spring 2011.

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Contents

1 Maxwell's Equations

The response fields $\overrightarrow{P}(\overrightarrow{r},t)$ and $\overrightarrow{M}(\overrightarrow{r},t)$ in non-dispersive matter can be related to the source fields $\overrightarrow{E}(\overrightarrow{r},t)$ and $\overrightarrow{H}(\overrightarrow{r},t)$ as follows.

$$\overrightarrow{P}(\overrightarrow{r},t) = \epsilon_0 \chi(\overrightarrow{r}) \overrightarrow{E}(\overrightarrow{r},t) \tag{1}$$

$$\overrightarrow{M}(\overrightarrow{r},t) = \chi_m(\overrightarrow{r}) \overrightarrow{H}(\overrightarrow{r},t) \tag{2}$$

This indicates linearity, in which the responses are linear functions of the inputs.

Following from these expressions, we can compute the fields as

$$\overrightarrow{D}(\overrightarrow{r},t) = \epsilon_0 \overrightarrow{E}(\overrightarrow{r},t) + \overrightarrow{P}(\overrightarrow{r},t) = \epsilon_0(1+\chi(\overrightarrow{r}))\overrightarrow{E}(\overrightarrow{r},t) = \epsilon(\overrightarrow{r})\overrightarrow{E}(\overrightarrow{r},t) \tag{3}$$

$$\overrightarrow{B}(\overrightarrow{r},t) = \mu_0(\overrightarrow{H}(\overrightarrow{r},t)+\overrightarrow{M}(\overrightarrow{r},t)) = \mu_0(1+\chi_m(\overrightarrow{r}))\overrightarrow{H}(\overrightarrow{r},t) = \mu(\overrightarrow{r})\overrightarrow{H}(\overrightarrow{r},t) \tag{4}$$

Maxwell's equations for linear, isotropic (output and input are aligned at all times), and non-dispersive media now become

$$\nabla \times \overrightarrow{E}(\overrightarrow{r},t) = -\frac{\partial \overrightarrow{B}(\overrightarrow{r},t)}{\partial t} \tag{5}$$

$$\nabla \times \left(\frac{\overrightarrow{B}(\overrightarrow{r},t)}{\mu(\overrightarrow{r})} \right) = \overrightarrow{J}(\overrightarrow{r},t) + \frac{\partial \left(\epsilon(\overrightarrow{r})\overrightarrow{E}(\overrightarrow{r},t) \right)}{\partial t} \tag{6}$$

$$\nabla \cdot \epsilon(\overrightarrow{r})\overrightarrow{E}(\overrightarrow{r},t) = \rho(\overrightarrow{r},t) \tag{7}$$

$$\nabla \cdot \overrightarrow{B}(\overrightarrow{r},t) = 0 \tag{8}$$

Assuming homogeneous media in which the material parameters are independent of position and are uniform throughout, the above equations can be further reduced to

$$\nabla \times \overrightarrow{E}(\overrightarrow{r},t) = -\frac{\partial \overrightarrow{B}(\overrightarrow{r},t)}{\partial t} \tag{9}$$

$$\nabla \times \overrightarrow{B}(\overrightarrow{r},t) = \mu \left(\overrightarrow{J}(\overrightarrow{r},t) + \frac{\partial \left(\epsilon \overrightarrow{E}(\overrightarrow{r},t) \right)}{\partial t} \right) \tag{10}$$

$$\nabla \cdot \overrightarrow{E}(\overrightarrow{r},t) = \frac{\rho(\overrightarrow{r},t)}{\epsilon} \tag{11}$$

$$\nabla \cdot \overrightarrow{B}(\overrightarrow{r},t) = 0 \tag{12}$$

The wave equations for these fields can be determined by applying $\nabla\times$ to both sides of the curl equations above, and using Vector Calculus to simplify the resulting expressions.

$$\nabla^2\overrightarrow{E}(\overrightarrow{r},t) - \mu\epsilon\frac{\partial^2\overrightarrow{E}(\overrightarrow{r},t)}{\partial t^2} = \frac{\nabla\rho}{\epsilon} + \mu\frac{\partial\overrightarrow{J}(\overrightarrow{r},t)}{\partial t} \tag{13}$$

$$\nabla^2\overrightarrow{B}(\overrightarrow{r},t) - \mu\epsilon\frac{\partial^2\overrightarrow{B}(\overrightarrow{r},t)}{\partial t^2} = -\mu\nabla\times\overrightarrow{J}(\overrightarrow{r},t) \tag{14}$$

Note that

1. For free space, the values of ϵ and μ are ϵ_0 and μ_0.
2. For non-conducting media, the charge and current densities are zero. Hence, the right sides of the wave equations reduce to zero.

2 Dispersive Media

For the case of dispersive media, the material characteristics are dependent not simply on position but also on time. That is,

$$\overrightarrow{P}(\overrightarrow{r},t) = \epsilon_0\chi(\overrightarrow{r},t) * \overrightarrow{E}(\overrightarrow{r},t) \tag{15}$$

$$\overrightarrow{M}(\overrightarrow{r},t) = \chi_m(\overrightarrow{r},t) * \overrightarrow{H}(\overrightarrow{r},t) \tag{16}$$

$$\overrightarrow{J}(\overrightarrow{r},t) = \sigma(\overrightarrow{r},t) * \overrightarrow{E}(\overrightarrow{r},t) \tag{17}$$

From these expressions, it immediately follows that

$$\overrightarrow{D}(\overrightarrow{r},t) = \epsilon(\overrightarrow{r},t) * \overrightarrow{E}(\overrightarrow{r},t) \tag{18}$$

$$\overrightarrow{B}(\overrightarrow{r},t) = \mu(\overrightarrow{r},t) * \overrightarrow{H}(\overrightarrow{r},t) \tag{19}$$

3 Wave Equations in the Frequency Domain

Frequency-domain expressions for the fields immediately follow by Fourier or phasor transformation. Fourier transformation can be applied to all fields, while phasor transformation is limited to time-harmonic fields.

The wave equations in the frequency domain are now the following.

$$\left[\nabla^2 + \omega^2\tilde{\mu}(\omega)\tilde{\epsilon}(\omega)\right]\overrightarrow{E}(\overrightarrow{r}) = \frac{\nabla\rho(\overrightarrow{r})}{\tilde{\epsilon}(\omega)} + j\omega\tilde{\mu}(\omega)\overrightarrow{J}(\overrightarrow{r}) \tag{20}$$

$$\left[\nabla^2 + \omega^2\tilde{\mu}(\omega)\tilde{\epsilon}(\omega)\right]\overrightarrow{H}(\overrightarrow{r}) = -\nabla\times\overrightarrow{J}(\overrightarrow{r}) \tag{21}$$

4 The Helmholtz Equation and Plane Waves

In the absence of free charge and current density, the frequency domain wave equation for $\overrightarrow{E}(\overrightarrow{r})$ becomes the mathematically homogeneous Helmholtz equation

$$\nabla^2 \overrightarrow{E}(\overrightarrow{r}) + k^2(\omega)\overrightarrow{E}(\overrightarrow{r}) = 0 \tag{22}$$

where

$$k^2(\omega) = \omega^2 \tilde{\mu}(\omega)\tilde{\epsilon}(\omega)\left(1 + \frac{\tilde{\sigma}(\omega)}{j\omega\tilde{\epsilon}(\omega)}\right) \tag{23}$$

For a suitably defined n(ω) such that

$$k^2(\omega) = \left(\frac{\omega}{c}\right)^2 n^2(\omega) = k_0^2 n^2(\omega) \tag{24}$$

For generally complex k(ω), it is reasonable to expect that n(ω) will be complex as well. Therefore, it can be expressed as

$$n(\omega) = n'(\omega) - jn''(\omega) \tag{25}$$

The simplest solutions of the Helmholtz equation are plane waves of the form

$$\overrightarrow{E}(\overrightarrow{r}) = \overrightarrow{E}_0 e^{-j\overrightarrow{k}\cdot\overrightarrow{r}} \tag{26}$$

A significant mathematical advantage of plane waves is that the replacement

$$\nabla \rightarrow -j\vec{k} \tag{27}$$

can be made. In addition, for plane waves,

$$\overrightarrow{k}(\omega) = [\beta(\omega) - j\alpha(\omega)]\vec{1}_k \tag{28}$$

For μ, ϵ, and σ independent of ω, the constants β and α can be determined from

$$2\beta^2 = \omega^2\mu\epsilon\left[1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right] \tag{29}$$

and

$$2\alpha^2 = \omega^2 \mu\epsilon \left[-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} \right] \tag{30}$$

In dispersive media, a frequency dependence is observed in the expressions for $\tilde{\chi}$, $\tilde{\chi_m}$ and $\tilde{\sigma}$.

5 Energy and Power Densities

The electric and magnetic energy densities are given by

$$u_e(\overrightarrow{r}, t) = \frac{1}{2}\overrightarrow{D}(\overrightarrow{r}, t) \cdot \overrightarrow{E}(\overrightarrow{r}, t) \tag{31}$$

$$u_m(\overrightarrow{r}, t) = \frac{1}{2}\overrightarrow{B}(\overrightarrow{r}, t) \cdot \overrightarrow{H}(\overrightarrow{r}, t) \tag{32}$$

The rate of change of energy stored in the electromagnetic field in volume V is given by

$$\frac{dU_e}{dt} + \frac{dU_m}{dt} = \iiint\limits_V \left\{ \frac{\partial u_e(\overrightarrow{r}, t)}{\partial t} + \frac{\partial u_m(\overrightarrow{r}, t)}{\partial t} \right\} dv \tag{33}$$

6 The Poynting Vector

The instantaneous Poynting vector is defined as

$$\overrightarrow{P}(\overrightarrow{r}, t) = \overrightarrow{E}(\overrightarrow{r}, t) \times \overrightarrow{H}(\overrightarrow{r}, t) \tag{34}$$

The instantaneous power through a surface S can be computed from the instantaneous Poynting vector as

$$P(t) = \iint\limits_S \overrightarrow{n} \cdot \overrightarrow{P}(\overrightarrow{r}, t) dS \tag{35}$$

The average power through S is computed over an interval T as

$$P_{av} = \frac{1}{T}\int\limits_0^T P(t)dt = \iint\limits_S \overrightarrow{n} \cdot \overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) dS \tag{36}$$

where

$$\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) = \frac{1}{T}\int\limits_0^T = \frac{1}{2}\mathcal{R}e\left\{\overrightarrow{E}(\overrightarrow{r}) \times \overrightarrow{H}^*(\overrightarrow{r})\right\} \tag{37}$$

7 Electrodynamic Potentials

The scalar potential $\Phi(\overrightarrow{r},t)$ and the vector potential $\overrightarrow{A}(\overrightarrow{r},t)$ are defined as follows.

$$\nabla\cdot\overrightarrow{B}(\overrightarrow{r},t)=0 \implies \overrightarrow{B}(\overrightarrow{r},t)=\nabla\times\overrightarrow{A}(\overrightarrow{r},t). \tag{38}$$

This follows directly from the fact that $\nabla\cdot$ $(\nabla\times \overrightarrow{\ })(\overrightarrow{r},t)$ $=0$ for an arbitrary vector field $\overrightarrow{\ }(\overrightarrow{r},t)$

$$\nabla\times\overrightarrow{E}(\overrightarrow{r},t)=-\frac{\partial\overrightarrow{B}(\overrightarrow{r},t)}{\partial t}=-\nabla\times\frac{\partial\overrightarrow{A}(\overrightarrow{r},t)}{\partial t} \tag{39}$$

Now, it follows that

$$\nabla\times\overrightarrow{E}(\overrightarrow{r},t)+\nabla\times\frac{\partial\overrightarrow{A}(\overrightarrow{r},t)}{\partial t}=0 \tag{40}$$

implying that

$$\overrightarrow{E}(\overrightarrow{r},t)=-\nabla\Phi(\overrightarrow{r},t)-\frac{\partial\overrightarrow{A}(\overrightarrow{r},t)}{\partial t} \tag{41}$$

The remaining Maxwell equations can be used to relate these electrodynamic potentials to the sources $\overrightarrow{J}(\overrightarrow{r},t)$ and $\rho(\overrightarrow{r},t)$ as follows.

From the curl of $\overrightarrow{B}(\overrightarrow{r},t)$,

$$\nabla\times\overrightarrow{B}(\overrightarrow{r},t)=\mu\left(\overrightarrow{J}(\overrightarrow{r},t)+\epsilon\frac{\partial\overrightarrow{E}(\overrightarrow{r},t)}{\partial t}\right) \tag{42}$$

But

$$\nabla\times\overrightarrow{B}(\overrightarrow{r},t)=\nabla\times\nabla\times\overrightarrow{A}(\overrightarrow{r},t)=\nabla\nabla\cdot\overrightarrow{A}(\overrightarrow{r},t)-\nabla^2\overrightarrow{A}(\overrightarrow{r},t) \tag{43}$$

Combining equations 42 and 43,

$$\nabla^2\overrightarrow{A}(\overrightarrow{r},t)-\mu\epsilon\frac{\partial^2\overrightarrow{A}(\overrightarrow{r},t)}{\partial t^2}=-\mu\overrightarrow{J}(\overrightarrow{r},t)+\nabla\left(\nabla\cdot\overrightarrow{A}(\overrightarrow{r},t)+\mu\epsilon\frac{\partial\Phi(\overrightarrow{r},t)}{\partial t}\right) \tag{44}$$

From the divergence of $\overrightarrow{E}(\overrightarrow{r},t)$,

$$\nabla\cdot\overrightarrow{E}(\overrightarrow{r},t)=\frac{\rho}{\epsilon} \tag{45}$$

But

$$\nabla \cdot \overrightarrow{E}(\overrightarrow{r}, t) = \nabla \cdot \left(-\nabla\Phi(\overrightarrow{r}, t) - \frac{\partial \overrightarrow{A}(\overrightarrow{r}, t)}{\partial t} \right) \tag{46}$$

Combining equations 45 and 46,

$$\nabla^2\Phi(\overrightarrow{r}, t) = -\frac{\rho}{\epsilon} - \frac{\partial \left(\nabla \cdot \overrightarrow{A}(\overrightarrow{r}, t) \right)}{\partial t} \tag{47}$$

We have determined the specification of $\nabla\times \overrightarrow{A}(\overrightarrow{r}, t)$ from the knowledge of $\overrightarrow{B}(\overrightarrow{r}, t)$. For full specification of $\overrightarrow{A}(\overrightarrow{r}, t)$ itself, we will need to know $\nabla\cdot \overrightarrow{A}(\overrightarrow{r}, t)$. This freedom to select $\nabla\cdot \overrightarrow{A}(\overrightarrow{r}, t)$ gives rise to several different potentials that generate the same $\overrightarrow{E}(\overrightarrow{r}, t)$ and $\overrightarrow{B}(\overrightarrow{r}, t)$ fields.

In the Lorenz Gauge, $\nabla\cdot \overrightarrow{A}(\overrightarrow{r}, t)$ is chosen such that

$$\nabla \cdot \overrightarrow{A}(\overrightarrow{r}, t) = -\mu\epsilon \frac{\partial \Phi(\overrightarrow{r}, t)}{\partial t} \tag{48}$$

while in the Coulomb Gauge, it is chosen such that

$$\nabla \cdot \overrightarrow{A}(\overrightarrow{r}, t) = 0 \tag{49}$$

These two special choices of result in solutions of different properties. The Coulomb Gauge, in particular, leads to a function $\Phi(\overrightarrow{r}, t)$ that obeys Poisson's equation but behaves like an unrealistic electrostatic potential that varies with time and responds instantaneously to changes in the source field $\rho(\overrightarrow{r}, t)$.

8 Unbounded Space Solution via Green's Theorem

If $\nabla^2 \overrightarrow{F}(\overrightarrow{r})$ is known at all points in space, $\overrightarrow{F}(\overrightarrow{r})$ can be determined from the volume integral

$$\overrightarrow{F}(\overrightarrow{r}) = \frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \frac{-\nabla'^2 \overrightarrow{F}(\overrightarrow{r}')}{R} \, dv' \tag{50}$$

in unbounded space.

9 Electrostatics

In electrostatics, the wave equation that was previously discussed for the electric field $\overrightarrow{E}(\overrightarrow{r})$ reduces to

$$\nabla^2 \overrightarrow{E}(\overrightarrow{r}) = \frac{\nabla \rho(\overrightarrow{r})}{\epsilon} \tag{51}$$

The unbounded space solution for the electric field $\overrightarrow{E}(\overrightarrow{r})$ can therefore be found using Green's theorem as shown in equation 50.

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \frac{-\nabla'^2 \overrightarrow{E}(\overrightarrow{r'})}{R} dv' = \frac{1}{4\pi\epsilon} \iiint_{-\infty}^{+\infty} \frac{-\nabla' \rho(\overrightarrow{r'})}{R} dv' = -\nabla \left(\frac{1}{4\pi\epsilon} \iiint_{-\infty}^{+\infty} \frac{\rho(\overrightarrow{r'})}{R} dv' \right) \tag{52}$$

10 Magnetostatics

In magnetostatics, the wave equation that was previously discussed for the magnetic field $\overrightarrow{H}(\overrightarrow{r})$ reduces to

$$\nabla^2 \overrightarrow{H}(\overrightarrow{r}) = -\nabla \times \overrightarrow{J}(\overrightarrow{r}) \tag{53}$$

The unbounded space solution for the magnetic field $\overrightarrow{H}(\overrightarrow{r})$ can therefore be found using Green's theorem as shown in equation 50.

$$\overrightarrow{H}(\overrightarrow{r}) = \frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \frac{\nabla' \times \overrightarrow{J}(\overrightarrow{r'})}{R} dv' = \nabla \times \left(\frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \frac{\overrightarrow{J}(\overrightarrow{r'})}{R} dv' \right) \tag{54}$$

11 Helmholtz Theorem and Helmholtz Potentials

The Helmholtz theorem helps us resolve a given field $\overrightarrow{F}(\overrightarrow{r})$ into its solenoidal (zero divergence) and irrotational (zero curl) components. As such, without proof, it can be shown using identity 50 that

$$\overrightarrow{F^L}(\overrightarrow{r}) = -\nabla\nabla \cdot \iiint_{-\infty}^{+\infty} \frac{\overrightarrow{F}(\overrightarrow{r}')}{4\pi R} dv' = -\nabla \Phi_F(\overrightarrow{r}) \tag{55}$$

$$\overrightarrow{F^T}(\overrightarrow{r}) = \nabla \times \nabla \times \iiint_{-\infty}^{+\infty} \frac{\overrightarrow{F}(\overrightarrow{r}')}{4\pi R}\, dv' = \nabla \times \overrightarrow{A}_F(\overrightarrow{r}) \tag{56}$$

In the above equations, $\overrightarrow{F^L}(\overrightarrow{r})$ and $\overrightarrow{F^T}(\overrightarrow{r})$ are the lamellar and transverse components of $\overrightarrow{F}(\overrightarrow{r})$, respectively.

$\Phi_F(\overrightarrow{r})$ and $\overrightarrow{A}_F(\overrightarrow{r})$ are known as the Helmholtz potentials of $\overrightarrow{F}(\overrightarrow{r})$.

From these expressions, it can be seen that for an irrotational vector field $\overrightarrow{F}(\overrightarrow{r})$, the transverse component disappears and only the lamellar component remains. Similarly, for an irrotational vector field $\overrightarrow{F}(\overrightarrow{r})$, the lamellar component disappears, leaving the transverse component only.

12 Solution of the Wave Equation in Unbounded Space

In previous sections, the solution to the inhomogeneous wave equation were found using Green's theorem and were used to determine a vector field $\overrightarrow{F}(\overrightarrow{r})$ from $\nabla^2 \overrightarrow{F}(\overrightarrow{r})$ in unbounded space. A more general approach to solving the wave equation in unbounded space is to consider the effects of an additional proportional term.

The solution $\overrightarrow{F}(\overrightarrow{r})$ of the wave equation

$$(\nabla^2 + k^2)\overrightarrow{F}(\overrightarrow{r}) = -\overrightarrow{f}(\overrightarrow{r}) \tag{57}$$

in unbounded space is

$$\overrightarrow{F}(\overrightarrow{r}) = \overrightarrow{f}(\overrightarrow{r}) *_3 e^{-jkr} = \iiint_{-\infty}^{+\infty} \overrightarrow{f}(\overrightarrow{r}') \frac{e^{-jk|\overrightarrow{r}-\overrightarrow{r}'|}}{4\pi|\overrightarrow{r}-\overrightarrow{r}'|}\, dv' \tag{58}$$

The potential integrals in unbounded space are therefore

$$\Phi(\overrightarrow{r}) = \frac{1}{4\pi\epsilon}\rho(\overrightarrow{r}) *_3 \frac{e^{-jkr}}{r} \tag{59}$$

$$\overrightarrow{A}(\overrightarrow{r}) = \frac{\mu}{4\pi}\overrightarrow{J}(\overrightarrow{r}) *_3 \frac{e^{-jkr}}{r} \tag{60}$$

The field integrals in unbounded space are similarly found from

$$\overrightarrow{H}(\overrightarrow{r}) = (\nabla \times \overrightarrow{J}(\overrightarrow{r})) *_3 \frac{e^{-jkr}}{4\pi r} \tag{61}$$

$$\overrightarrow{E}(\overrightarrow{r}) = -\left(\frac{\nabla\rho(\overrightarrow{r})}{\epsilon} + j\omega\mu\overrightarrow{J}(\overrightarrow{r})\right) *_3 \frac{e^{-jkr}}{4\pi r} \tag{62}$$

13 Time-Dependent Field Satisfying D'Alembert's Equation

The previous sections dealt with finding the solution $\overrightarrow{F}(\overrightarrow{r})$ of the equation

$$(\nabla^2 + k^2)\overrightarrow{F}(\overrightarrow{r}) = -\overrightarrow{f}(\overrightarrow{r}) \tag{63}$$

We now determine the time-dependent field $\overrightarrow{F}(\overrightarrow{r}, t)$ from the phasor or Fourier-transformed field $\overrightarrow{F}(\overrightarrow{r})$.

$$\overrightarrow{F}(\overrightarrow{r}) = \overrightarrow{f}(\overrightarrow{r}) *_3 \frac{e^{-jk\frac{r}{v}}}{4\pi r} \tag{64}$$

Applying the inverse Fourier transform to $\overrightarrow{F}(\overrightarrow{r})$,

$$\overrightarrow{F}(\overrightarrow{r}, t) = \mathcal{F}^{-1}\{\overrightarrow{F}(\overrightarrow{r})\} = \overrightarrow{f}(\overrightarrow{r}, t) *_4 \frac{\delta\left(t - \frac{r}{v}\right)}{4\pi r} \tag{65}$$

where $*_4$ indicates a three-dimensional spatial and one-dimensional temporal convolution. Note the time retardation involved above.

This final result has significant implications. It tells us that the response field $\overrightarrow{F}(\overrightarrow{r}, t)$ at the observation point $(\overrightarrow{r}, t)$ at time t is the sum of the contributions made by all the elements of the source field $\overrightarrow{f}(\overrightarrow{r}, t)$.

In summary, $\overrightarrow{B}(\overrightarrow{r})$ and $\overrightarrow{E}(\overrightarrow{r})$ can be determined from the source $\overrightarrow{J}(\overrightarrow{r})$ directly, or by the use of the potentials in unbounded space.

14 Far Field Derivatives of Retarded Potentials

In the far-field approximations, one can observe, by using the potentials in the Lorenz Gauge, that ∇ acts on retarded potentials the same way that $-jk\overrightarrow{1_r}$ would. For the time-dependent potentials, ∇ has the same effect as the operator $-\frac{1}{v}\frac{\partial}{\partial t}\overrightarrow{1_r}$. Using these newly found operators to find $\overrightarrow{E}(\overrightarrow{r})$ and $\overrightarrow{B}(\overrightarrow{r})$ in the far field, we note that both fields do not have radial components. Instead, they are both transverse to $\overrightarrow{1_r}$. With $\overrightarrow{1_r}$, they form an orthogonal triplet.

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{\sqrt{\mu\epsilon}\overrightarrow{B}(\overrightarrow{r}) \times \overrightarrow{1_r}} = \sqrt{\frac{\mu}{\epsilon}}\overrightarrow{H}(\overrightarrow{r}) \times \overrightarrow{1_r} \tag{66}$$

$$\frac{1}{\sqrt{\mu\epsilon}}\overrightarrow{B}(\overrightarrow{r}) = \sqrt{\frac{\mu}{\epsilon}}\overrightarrow{H}(\overrightarrow{r}) = \overrightarrow{1_r} \times \overrightarrow{E}(\overrightarrow{r}) \tag{67}$$

15 Far Zone Potentials and Fields Related to the Radiation Vector $\overrightarrow{N}$

The radiation vector is another way of defining a field. It can be computed from the current density $\overrightarrow{J}(\overrightarrow{r})$ as

$$\overrightarrow{N}(\theta, \phi) = \iiint_{-\infty}^{+\infty} \overrightarrow{J}(\overrightarrow{r}')e^{jk\overrightarrow{1_r(\theta,\phi)}\cdot\overrightarrow{r}'}dv' \tag{68}$$

$$\overrightarrow{A}(\overrightarrow{r}) = \mu\frac{e^{-jkr}}{4\pi r}\overrightarrow{N}(\theta, \phi) \tag{69}$$

$$\Phi(\overrightarrow{r}) = \sqrt{\frac{\mu}{\epsilon}}\frac{e^{-jkr}}{4\pi r}\overrightarrow{N}(\theta, \phi) \cdot \overrightarrow{1_r} \tag{70}$$

$$\overrightarrow{B}(\overrightarrow{r}) = jk\mu\frac{e^{-jkr}}{4\pi r}\overrightarrow{N}(\theta, \phi) \times \overrightarrow{1_r} \tag{71}$$

$$\overrightarrow{E}(\overrightarrow{r}) = -j\omega\mu\frac{e^{-jkr}}{4\pi r}(\overrightarrow{N}(\theta, \phi) \times \overrightarrow{1_r}) \times \overrightarrow{1_r} \tag{72}$$

The average Poynting Vector $\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r})$ can be determined from the usual expression

$$\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) = \frac{1}{2}Re\{\overrightarrow{E}(\overrightarrow{r}) \times \overrightarrow{H}(\overrightarrow{r})^*\} \tag{73}$$

Simplifications of this expression using the above equations for the potentials leads to

$$\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) = \overrightarrow{1_r}\frac{\omega^2}{2\eta}|\overrightarrow{A}(\overrightarrow{r}) \times \overrightarrow{1_r}|^2 \tag{74}$$

The explicit dependence of the average Poynting Vector on the radiation vector can be shown to be

$$\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) = \overrightarrow{1_r}\frac{\eta k^2}{32\pi^2}\frac{|\overrightarrow{N}(\theta,\phi)\times\overrightarrow{1_r}|}{r^2} \tag{75}$$

16 Moments of a Charge Distribution

Fixed charges can be characterized by their moments as their distributions vary with time. The monopole, dipole, and quadrupole moments are calculated as follows.

$$q = \iiint_{-\infty}^{+\infty} \rho(\overrightarrow{r}, t)dv \tag{76}$$

$$\overrightarrow{p}(t) = \iiint_{-\infty}^{+\infty} \rho(\overrightarrow{r}, t)\overrightarrow{r}dv \tag{77}$$

$$\overleftrightarrow{Q}(t) = \iiint_{-\infty}^{+\infty} \rho(\overrightarrow{r}, t)\overrightarrow{r}\overrightarrow{r}dv \tag{78}$$

17 Moments of a Current Density

Similar to the moments of charge distributions, we can define moments for current density. The most commonly used moments are the monopole moment and the dyadic magnetic dipole moment.

$$\text{monopole moment} = \iiint_{-\infty}^{+\infty} \overrightarrow{J}(\overrightarrow{r}, t)dv \tag{79}$$

$$\text{dipole magnetic moment} = \iiint_{-\infty}^{+\infty} \overrightarrow{J}(\overrightarrow{r}, t)\overrightarrow{r}dv \tag{80}$$

It can be shown that the monopole moment is further found from the derivative of the dipole charge distribution moment.

$$\iiint_{-\infty}^{+\infty} \overrightarrow{J}(\overrightarrow{r}, t)dv = \frac{d\overrightarrow{p}(t)}{dt} \tag{81}$$

The vector magnetic dipole moment can be determined from the source $\overrightarrow{J}(\overrightarrow{r},t)$ as

$$\overrightarrow{m}(t) = \frac{1}{2}\iiint_{-\infty}^{+\infty} \overrightarrow{r} \times \overrightarrow{J}(\overrightarrow{r},t)dv \tag{82}$$

$$\overleftrightarrow{m}(t) = \frac{1}{2}\frac{d\overleftrightarrow{Q}(t)}{dt} + \overrightarrow{m}(t) \times \overleftrightarrow{I}(t) \tag{83}$$

All dyadics can be written as sums of symmetric and antisymmetric parts. $\overleftrightarrow{m}(t)$ can be split into these parts from the above expression.

$$\overleftrightarrow{m}(t)_{symmetric} = \frac{1}{2}\frac{d\overleftrightarrow{Q}(t)}{dt} \tag{84}$$

$$\overleftrightarrow{m}(t)_{antisymmetric} = \overrightarrow{m}(t) \times \overleftrightarrow{I}(t) \tag{85}$$

18 Determination of $\rho(\overrightarrow{r},t)$ and $\overrightarrow{J}(\overrightarrow{r},t)$ from the Moments

The idea of using moments to recover the original distribution is heavily used in Probability. We can use a similar idea to recover the source fields from their known moments.

$$\rho(\overrightarrow{r},t) \approx q\delta(\overrightarrow{r}) - \overrightarrow{p}(t)\cdot\nabla\delta(\overrightarrow{r}) + \frac{1}{2}\overleftrightarrow{Q}(t) : \nabla\nabla\delta(\overrightarrow{r}) + ... \tag{86}$$

$$\overrightarrow{J}(\overrightarrow{r},t) = \frac{d\overrightarrow{p}(t)}{dt}\delta(\overrightarrow{r}) - \overleftrightarrow{m}(t)\cdot\nabla\delta(\overrightarrow{r}) + ... \tag{87}$$

The phasor- and Fourier-transformed values of $\rho(\overrightarrow{r},t)$ and $\overrightarrow{J}(\overrightarrow{r},t)$ follow directly from these expressions. They can also be computed directly from the phasor- and Fourier-transformed moments due to linearity.

19 Hertzian Dipole

The Hertzian dipole is one whose current density is defined as

$$\overrightarrow{J}(\overrightarrow{r},t) = \vec{1}_z\delta(x)\delta(y)i(z,t) \tag{88}$$

where

$$i(z,t) = \Pi\left(\frac{z}{l}\right) i(t) \tag{89}$$

From these expressions,

$$\frac{d\overrightarrow{p}(t)}{dt} = \vec{1}_z i(t) l \tag{90}$$

and

$$\overleftrightarrow{m}(t) = 0 \tag{91}$$

20 Hertzian Point Dipole

The Hertzian point dipole is a special case of the Hertzian dipole in which

$$i(z,t) = i(t) l \delta(z) \tag{92}$$

Hence, the expression for the current density is

$$\overrightarrow{J}(\overrightarrow{r}, t) = \vec{1}_z i(t) l \delta(\overrightarrow{r}) \tag{93}$$

21 Multipole Moment Expansion of the Radiation Vector $\overrightarrow{N}(\theta, \phi)$

Since the evaluation of the integral which relates the radiation vector $\overrightarrow{N}(\theta, \phi)$ to the current density $\overrightarrow{J}(\overrightarrow{r}, t)$ may be difficult to evaluate, an alternative way of determining it is through the use of the moments described in earlier sections.

$$\overrightarrow{N}(\theta, \phi) = j\omega\vec{p} + jk\frac{j\omega}{2}\tilde{Q} \cdot \vec{1}_r + jk\vec{m} \times \vec{1}_r + ... \tag{94}$$

22 Straight Wire Antennas

For a straight wire antenna,

$$\overrightarrow{J}(\overrightarrow{r}) = \vec{1}_z J(\overrightarrow{r}) \tag{95}$$

From this, the expression for the radiation vector immediately follows.

$$\overrightarrow{N}(\theta, \phi) = \vec{1}_z N_z(\theta, \phi) \tag{96}$$

Note that N_z is independent of ϕ since $\vec{J}$ is independent of ϕ. $\overrightarrow{H}(\overrightarrow{r})$, $\overrightarrow{E}(\overrightarrow{r})$, and $\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r})$ follow immediately from these results.

23 Thin Wire Antennas

For a thin wire antenna of length l = 2h,

$$\overrightarrow{J}(\overrightarrow{r}) = \vec{1}_z I(z)\delta(x)\delta(y)\Pi\left(\frac{z}{l}\right) \tag{97}$$

It can be seen that

$$\overrightarrow{N}(\theta,\phi) = \vec{1}_z N(\theta) \tag{98}$$

and the radiation vector is independent of ϕ due to cylindrical symmetry of $\overrightarrow{J}(\overrightarrow{r})$.

Given I(z) as

$$I(z) = I_m sin(k(h - |z|)) \tag{99}$$

one can calculate this radiation vector to be

$$N(\theta) = \frac{2I_m}{k}\frac{F(\theta;kh)}{sin\theta} \tag{100}$$

where

$$F(\theta;kh) = \frac{cos(khcos(\theta)) - cos(kh)}{sin(\theta)} \tag{101}$$

$F(\theta;kh)$ is known as the **radiation pattern** of the antenna. The antenna is said to be **omnidirectional** due to the lack of dependence of ϕ in the above expressions.

For this radiation pattern, the fields and their potentials can be calculated as

$$\overrightarrow{A}(\overrightarrow{r}) = \mu\vec{N}(\theta)\frac{e^{-jkr}}{4\pi r} \tag{102}$$

$$\overrightarrow{H}(\overrightarrow{r}) = \vec{1}_\phi j2I_m F(\theta)\frac{e^{-jkr}}{4\pi r} \tag{103}$$

$$\overrightarrow{E}(\overrightarrow{r}) = \vec{1}_\theta \eta H_\phi \tag{104}$$

$$\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) = \vec{1}_r\sqrt{\frac{\mu}{\epsilon}}\frac{I_m^2 F^2(\theta)}{8\pi^2 r^2} \tag{105}$$

$$P(\theta, \phi) = \sqrt{\frac{\mu}{\epsilon}} \frac{I_m^2 F^2(\theta)}{8\pi^2} \quad (106)$$

24 Radiated Power and Radiation Resistance

The radiated power P_T is defined as

$$P_T =_s \overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) \cdot \vec{n} ds \quad (107)$$

It is the measure of average radiated power, commonly termed as transmitted power.

For the thin dipole, it is

$$P_T = \sqrt{\frac{\mu}{\epsilon}} \frac{I_m^2}{4\pi} \int_0^{\pi} F^2(\theta; kh) sin(\theta) d\theta \quad (108)$$

The expression under the integral evaluates to be a function of kh. It is usually rewritten as f(2kh) such that

$$P_T = \sqrt{\frac{\mu}{\epsilon}} \frac{I_m^2}{4\pi} f(2kh) \quad (109)$$

The radiation resistance referred to z = h - $\lambda_{\overline{4 where I(z) = I_m is}}$

$$R_{rad} = \frac{2P_T}{I_m^2} = \sqrt{\frac{\mu}{\epsilon}} \frac{f(2kh)}{2\pi} \quad (110)$$

For free space, the radiation resistance is 60f(2kh).

The driving point resistance at z = 0 can be found from R_{rad} as

$$R_{rad_{drivingpoint}} = \frac{R_{rad}}{sin^2(kh)} \quad (111)$$

Using sine and cosine integrals, a more approachable expression for f(2kh) may be found as

$$f(2kh) = (1+cos(2kh))cin(2kh) - \frac{1}{2}cos(2kh)cin(4kh) + \frac{1}{2}sin(2kh)si(4kh) - sin(2kh)si(2kh)$$

Note that

$$2kh = \frac{l}{\lambda} 2\pi \quad (113)$$

Letting

$$l = n\frac{\lambda}{2} \Rightarrow 2kh = n\pi \tag{114}$$

$f(n\pi)$ now becomes

$$f(n\pi) = \begin{cases} \frac{1}{2}cin(2n\pi) & \text{for even n} \\ f(n\pi) = 2cin(n\pi) - \frac{1}{2}cin(2n\pi) & \text{for odd n} \end{cases} \tag{115}$$

Numerical tables can be used to determine the values of the sine and cosine integrals for different values of n, the number of half-wavelengths.

25 Selected Antenna Parameters

25.1 Radiation Intensity

The radiation intensity is measured in Watts/Steradians and is computed from the average Poynting vector as

$$P(\theta, \phi) = \overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r}) \cdot \vec{1}_r r^2 \tag{116}$$

25.2 Radiated Power

The radiated power is measured in Watts and is computed as

$$P_{rad} = P(\theta, \phi) d\Omega \tag{117}$$

The radiation intensity of an isotropic antenna is

$$P_0 = \frac{P_{rad}}{4\pi} \tag{118}$$

25.3 Directional Gain

The directional gain $D(\theta, \phi)$ is computed as

$$D(\theta, \phi) = 4\pi \frac{P(\theta, \phi)}{P_T} \tag{119}$$

and measures the extent by which the radiation intensity in the direction of (θ, ϕ) differs from that of isotropic radiation.

25.4 Directivity

The directivity is the maximum directional gain and is obtained from the expression

$$D_0 = 4\pi \frac{P(\theta_0, \phi_0)}{P_T} \tag{120}$$

where $P(\theta_0, \phi_0)$ is the maximum radiation intensity.

25.5 Efficiency

The efficiency can be computed from the radiation and loss resistances as

$$\mathcal{E} = \frac{R_{rad}}{R_{rad} + R_{loss}} \tag{121}$$

25.6 Gain Function

The gain function is the ratio of the radiation intensity and the maximum intensity from a reference antenna with the same input power. It is computed from

$$G(\theta, \phi) = \mathcal{E} D(\theta, \phi) \tag{122}$$

25.7 Maximum Gain

The maximum gain is computed from

$$G = \mathcal{E} D_0 \tag{123}$$

26 Effective Aperture of an Antenna

A receiving antenna may be thought of as having a capture area such that the product of this area with the magnitude of the average Poynting vector gives P_{Rout}, the power delivered to the load.

$$A_e(\theta, \phi) = \frac{P_{Rout}}{|\overrightarrow{\mathcal{P}}_{av}(\overrightarrow{r})|} = \frac{2\eta P_{Rout}}{|\overrightarrow{E}(\overrightarrow{r})|^2} \tag{124}$$

This effective area, or capture area, may be related to the gain and directivity as follows.

$$A_e(\theta,\phi) = G(\theta,\phi)\frac{\lambda^2}{4\pi} = \mathcal{E}D(\theta,\phi)\frac{\lambda^2}{4\pi} \tag{125}$$

The maximum capture area may be computed from

$$A_{em} = A_e(\theta_0,\phi_0) = \mathcal{E}D_0\frac{\lambda^2}{4\pi} = D_0\frac{\lambda^2}{4\pi} \tag{126}$$

where the efficiency has been assumed to be 100% for the last conclusion above.

27 Friis Transmission Formulas

Given a transmitting antenna with input power $P_{T_{in}}$ and a receiving antenna with an output power $P_{R_{out}}$,

$$\frac{P_{R_{out}}}{P_{T_{in}}} = G_T(\theta_T,\phi_T)G_R(\theta_R,\phi_R)\left(\frac{\lambda}{4\pi r}\right)^2 \tag{127}$$

In terms of the capture area, this can be expressed as

$$\frac{P_{R_{out}}}{P_{T_{in}}} = \frac{A_{eT}(\theta_T,\phi_T)A_{eR}(\theta_R,\phi_R)}{(\lambda r)^2} \tag{128}$$

Taking dissipation into account, the ratio of the available input power to the transmitted power P_T is

$$\frac{P_R}{P_T} = \frac{A_{eT}(\theta_T,\phi_T)A_{eR}(\theta_R,\phi_R)}{\mathcal{E}_T\mathcal{E}_R(\lambda r)^2} \tag{129}$$

Note that for lossless antennas,

$$\frac{P_{R_{out}}}{P_{T_{in}}} = \frac{P_R}{P_T} \tag{130}$$

28 Scalar Theory of Diffraction

The scalar theory of diffraction treats the optical field as a scalar disturbance u(x,y,z) that satisfies the wave equation

$$(\nabla^2 + k^2)u(x,y,z) = 0 \tag{131}$$

The problem essentially leads to a solution of u(x,y,z) from known u(x,y,0).

The plane wave $e^{-j\vec{k}\cdot\vec{r}}$ is a solution to this equation. The general solution consists of a superposition of plane waves of various amplitudes such that

$$u_z(x,y) = \iint_{-\infty}^{+\infty} g(2\pi\nu_x, 2\pi\nu_y) e^{-j2\pi(x\nu_x+y\nu_y+z\nu_z)} d\nu_x d\nu_y \tag{132}$$

It is easy to see that

$$u_0(x,y) = u_z(x,y)|_{z=0} \leftrightarrow g(2\pi\nu_x, 2\pi\nu_y) \tag{133}$$

This Fourier pair suggests a system-like analysis of the disturbance. Indeed,

$$u_z(x,y) = u_0(x,y) * *h_z(x,y) \tag{134}$$

where

$$h_z(x,y) = \iint_{-\infty}^{+\infty} e^{-j2\pi(x\nu_x+y\nu_y+z\nu_z)} d\nu_x d\nu_y \tag{135}$$

$h_z(x,y)$ is known as the **impulse response of free space**, or the **point spread function**.

The result of the above integration yields

$$h_z(x,y) = \left(\frac{1}{r} + jk\right) \frac{z}{r} \frac{e^{-jkr}}{2\pi r} \tag{136}$$

Given this expression for $h_z(x,y)$, the **Rayleigh-Sommerfeld diffraction formula** tells us that

$$u_z(x,y) = \iint_{-\infty}^{+\infty} u_0(x',y') \left(\frac{1}{R_0} + jk\right) \frac{z}{R_0} \frac{e^{-jkR_0}}{2\pi R_0} dx'dy' \tag{137}$$

At optical frequencies,

$$u_z(x,y) \approx \iint_{aperture} u_0(x',y') \left(\frac{j}{\lambda}\right) \frac{z}{R_0} \frac{e^{-jkR_0}}{R_0} dx'dy' \tag{138}$$

A further approximation made by Fresnel is to turn the obliquity factor into unity, such that

$$u_z(x,y) \approx \left(\frac{j}{\lambda}\right) \frac{e^{-jkz}}{z} \iint_{aperture} u_0(x',y') e^{-j\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'dy' \tag{139}$$

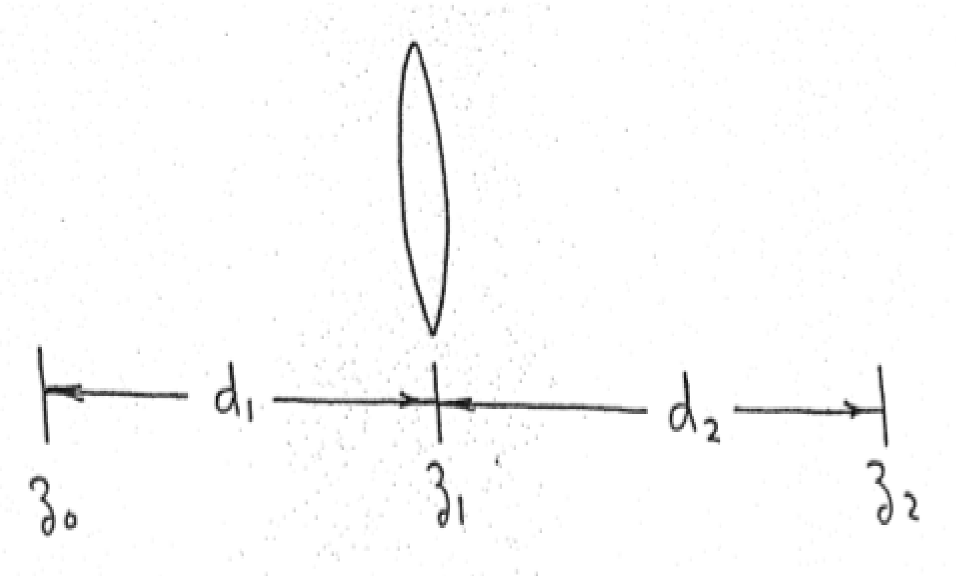

Figure 1: **Single Lens System with Front and Back Translation Distances d_1 and d_2**

An additional approximation, known as the **Fraunhoffer Approximation**, destroys the convolution that exists between the system impulse response and the input at z = 0. The resulting approximation is

$$u_z(x,y) \approx h_z(x,y) \iint\limits_{-\infty}^{+\infty} u_0(x',y') e^{j2\pi\left(\frac{x}{\lambda z}x' + \frac{y}{\lambda z}y'\right)} dx'dy' \tag{140}$$

It can be seen that there is a relationship between $u_z(x,y)$ and $u_0(x,y)$. This implies that the system behaves as a **spectrum analyzer**. In other words, propagation of a signal in free space for a sufficient distance results in spectral analysis with amplitude diminution.

29 Impulse Response of a Single Lens System

In the Fresnel approximation, the impulse response of a single lens system composed of a lens with free space in front of and behind it can be determined as (see figure on page 23)

$$h_{z_2-z_0}(x,y;\xi,\eta) = [h_{z_1-z_0}(x-\xi,y-\eta)t_{z_1}(x,y) ** h_{z_2-z_1}(x,y)] \tag{141}$$

Given the transmission function of the lens to be

$$t_z(x, y) = p(x, y)e^{j\phi_0}e^{-j\frac{k}{2f}(x^2+y^2)} \tag{142}$$

where f is the focal length of the lens, and p(x,y) is the pupil function describing the lens aperture, the expression for the system impulse response now becomes a lengthy expression shown below.

$$\iint_{-\infty}^{+\infty} p(x_1, y_1)e^{j\frac{k}{2}(\frac{1}{d_1}+\frac{1}{d_2}-\frac{1}{f})(x_1^2+y_1^2)}e^{-jk\left[\left(\frac{x_0}{d_1}+\frac{x_2}{d_2}\right)x_1+\left(\frac{y_0}{d_1}+\frac{y_2}{d_2}\right)y_1\right]}dx_1dy_1 \tag{143}$$

Note that the pupil function reduces to p(x,y) = 1 for a lens of infinite extent. This unrealistic assumption can be used to make further conclusions about the imaging condition and results from different values of the front and back distances d_1 and d_2.